REMARKS ON CONTRACTIVE PROJECTIONS IN L_p-SPACES

BY

L. TZAFRIRI*

ABSTRACT

The aim of this note is to study the structure of the range of a contractive projection in a non-separable L_p-space; $1 \leqq p < +\infty$.

A. Grothendieck [4] has proved that if a projection in an L_1-space is contractive, i.e. it has norm $\leqq 1$, then its range is isometric to an L_1-space. Later, under the assumption that (Ω, Σ, m) is a finite measure space, R. G. Douglas [2] has given a complete characterization of contractive projections in $L_1(\Omega, \Sigma, m)$ related closely to the notion of conditional expectation. Douglas' results have been extended by T. Ando [1] for L_p-spaces; $1 < p < +\infty$; he has proven that a contractive projection in an L_p-space; $1 < p < \infty$ over a finite measure space is similar to a conditional expectation and hence, its range is isometric to an L_p-space, Obviously, the range of a contractive projection in a separable L_p-space has the same structure.

The purpose of this note is to show that neither the separability of the space nor the condition imposed on the measure to be finite (or σ-finite) are essential and consequently to prove that the range of contractive projection in *any* L_p-space; $1 \leqq p < +\infty$ is isometric to an $-L_p$-space.

This result will be used to complete a characterization of L_p-spaces in terms of $\mathscr{L}_{p,\lambda}$-spaces introduced recently by J. Lindenstrauss and A. Pełczyński [5].

We shall need first some measure-theoretical results.

LEMMA 1. *Let (Ω, Σ, μ) be a measure space, $\mathscr{M}$ a separable subspace of $L_p(\Omega, \Sigma, \mu)$; $1 \leqq p < \infty$ and T a linear bounded operator on $L_p(\Omega, \Sigma, \mu)$. Then there exist a set $\tilde{\Omega} \in \Sigma$ and a sub σ-ring $\tilde{\Sigma}$ of Σ restricted to $\tilde{\Omega}$ such that:*

---

* Supported by National Science Foundation Grant GP–7475

Received January 8, 1969.

a) *$L_p(\tilde{\Omega},\tilde{\Sigma},\mu)$ is separable and hence $(\tilde{\Omega},\tilde{\Sigma},\mu)$ is σ-finite.*

b) $\mathscr{M} \subset L_p(\tilde{\Omega},\tilde{\Sigma},\mu)$.

c) *$L_p(\tilde{\Omega},\tilde{\Sigma},\mu)$ is an invariant subspace for T.*

Proof. Assume $\{f_n\}$ is a sequence dense in $\mathscr{M}$ and

$$g_n = \sum_{i=1}^{m_n} a_i^{(n)} \chi_{A_i^{(n)}}$$

is a sequence of simple functions whose closure contains f_n; $n = 1,2,\cdots$. By [3] Lemma III–8–4 the subring $\mathscr{B}_1$ generated by the sets $A_i^{(n)}$; $n = 1,2,\cdots$; $i = 1\ 2,\cdots,m_n$ is countable. Denote $\mathscr{C}_1 = \{T\chi_A \mid A \in \mathscr{B}_1\}$. Suppose now that the subring $\mathscr{B}_k$ and the set of functions $\mathscr{C}_k$ have been constructed and $\mathscr{B}_k$ is countable. Let

$$h_n = \sum_{i=1}^{p_n} b_i^{(n)} \chi_{B_i^{(n)}}$$

be a sequence of simple functions whose closure in $L_p(\Omega,\Sigma,\mu)$ contains $\mathscr{C}_k$. Then $\mathscr{B}_{k+1}$ will be defined as the (countable) subring generated by $\mathscr{B}_k$ and the sets $B_i^{(n)}$; $n = 1,2,\cdots,$; $i = 1,2\cdots,\ p_n$ and $\mathscr{C}_{k+1} = \{T\chi_A \mid A \in \mathscr{B}_{k+1}\}$.

Continuing so we obtain an increasing sequence of countable subrings $\mathscr{B}_1 \subset \mathscr{B}_2 \subset \cdots \subset \mathscr{B}_k \subset \cdots$ and an increasing sequence of collections of functions $\mathscr{C}_1 \subset \mathscr{C}_2 \subset \ldots \subset \mathscr{C}_k \subset \cdots$. Set $\mathscr{B} = \bigcup_{k=1}^{\infty} \mathscr{B}_k$, $\tilde{\Omega} = \bigcup_{A\in\mathscr{B}} A$ and let $\tilde{\Sigma}$ be the sub-σ-ring generated by $\mathscr{B}$. Since $\mathscr{B}$ is a countable subring the simple functions over $\mathscr{B}$ are dense in $L_p(\tilde{\Omega},\tilde{\Sigma},\mu)$ (cf. [3] Lemma III–8–3) and thus $L_p(\tilde{\Omega},\tilde{\Sigma},\mu)$ is separable. Evidently $\mathscr{B}_1 \subset \tilde{\Sigma}$ implies condition b. To prove that condition c holds let us remark that if $A \in \mathscr{B}_k$ for some k then $T\chi_A \in \mathscr{C}_k$ and therefore $T\chi_A$ can be approximated (in L_p-norm) by simple functions over $\mathscr{B}_{k+1} \subset \tilde{\Sigma}$. It follows that T maps simple functions over $\mathscr{B}$ into $L_p(\tilde{\Omega},\tilde{\Sigma},\mu)$ and this completes the proof. Q.E.D.

REMARK. For the particular case in which T is the identity operator Lemma 1 is proved in [3] Lemma III–8–5.

LEMMA 2. *Let (Ω',Σ',μ') be a σ-finite measure space. Then any closed subspace $\mathscr{M}$ of $L_p(\Omega',\Sigma',\mu')$; $1 \leqq p < +\infty$ contains a function with maximum support.*

Proof. This result has been proved by T. Ando [1] Lemma 3 for finite measures and can be extended to our case since $L_p(\Omega',\Sigma',\mu')$ is isometric to a space $L_p(\Omega',\Sigma',\mu'')$ ((where $\mu''(A) = \int_A \phi(\omega)\mu'(d\omega)$; $A \in \Sigma'$ for some function $\phi \in L_1(\Omega',\Sigma',\mu')$ satisfying $0 < \phi(\omega) \leqq 1$; $\omega \in \Omega'$) with μ'' being a finite measure and under this isometry the support is preserved. Q.E.D.

LEMMA 3. *Let P be a contractive projection in $L_p(\Omega,\Sigma,\mu)$; $1 \leqq p \neq 2 < +\infty$. If $Pg = g$ for some $g \in L_p(\Omega,\Sigma,\mu)$ and Σ' denotes the restriction of Σ to $\Omega' = S(g)$ then $L_p(\Omega',\Sigma',\mu)$ is invariant under P and the measure space (Ω',Σ',μ) is σ-finite.*

Proof. Consider a function $h \in L_p(\Omega',\Sigma',\mu)$ and let $\mathscr{M}$ be the (finite-dimensional) subspace generated bv g, h and Ph. Using Lemma 1 for the subspace $\mathscr{M}$ and the operator P we obtain a separable subspace $L_p(\tilde{\Omega},\tilde{\Sigma},\mu)$ of $L_p(\Omega,\Sigma,\mu)$ which is invariant under P and contains g, h and Ph. Let u be a function with maximum support in $PL_p(\tilde{\Omega},\tilde{\Sigma},\mu)$. Such a function exists in view of Lemma 2 since $L_p(\tilde{\Omega},\tilde{\Sigma},\mu)$ is σ-finite. Now let us set $G = S(u)$, $v(A) = \int_A |u(\omega)|^p \mu(d\omega)$ where A belongs to the restriction $\tilde{\Sigma}_G$ of $\tilde{\Sigma}$ to G. The operator Q in $L_p(G,\tilde{\Sigma}_G, v)$ defined by

$$Qv = \frac{P(uv)}{u}$$

is obviously a contractive projection which leaves invariant the function $1 \in L_p(G,\tilde{\Sigma}_G, v)$. By T. Ando [1] Theorem 1 and R. G. Douglas [2] Corollary 1 Q is a conditional expectation relative to some sub σ-field Σ_u of $\tilde{\Sigma}_G$.

Thus $Q(h/u) = Ph/u$ and $Q(g/u) = Pg/u = g/u$ are Σ_u-measurable functions, in particular $S(g) \in \Sigma_u$. It follows that

$$\int_A Q\left(\frac{h}{u}\right)|u|^p d\mu = \int_A \frac{h}{u}|u|^p d\mu = 0$$

for every $A \in \Sigma_u$; $A \subset S(u) - S(g)$. Hence $Ph/u = Q(h/u) = 0$ outside $S(g)$ and and this completes the proof. Q.E.D.

LEMMA 4. *Let (Ω,Σ,μ) be a general measure space and P a contractive projection in $L_p(\Omega,\Sigma,\mu)$; $1 \leqq p \neq 2 < +\infty$. For every $f \in L_p(\Omega,\Sigma,\mu)$ satisfying $Pf = f$ define*

$$X_f = \{P(hf) \mid hf \in L_p(\Omega,\Sigma,\mu)\}$$

Then X_f is a closed subspace of the range of P isometric to $L_p(S(f),\Sigma_f, |f|^p d\mu)$ where Σ_f is a sub-σ-field of $\Sigma_{S(f)}$ (the restriction of Σ to $S(f)$). Moreover, the unctions $f \cdot \sum_{i=1}^m a_i\chi_{\delta_i}$; $\delta_i \in \Sigma_f$ are dense in X_f.

Proof. Consider $f \in L_p(\Omega, \Sigma, \mu)$ satisfying $Pf = f$ and define

$$Q_f h = \frac{P(hf)}{f}$$

In view of Lemma 3, Q_f is a contractive projection in $L_p(S(f), \Sigma_{S(f)}, |f|^p d\mu)$ for which $Q_f 1 = 1$. Since $|f|^p d\mu$ is a finite measure, by T. Ando [1] Theorem 1 and R. G. Douglas [2] Corollary 1, Q_f will be a conditional expectation relative to a sub-σ-field Σ_f of $\Sigma_{S(f)}$ and therefore $Q_f L_p(S(f), \Sigma_{S(f)}, |f|^p d\mu) = L_p(S(f), \Sigma_f, |f|^p d\mu)$. The correspondence $Q_f h \leftrightarrow f Q_f h = P(hf)$ establishes an isometry between $L_p(S(f), \Sigma_f, |f|^p d\mu)$ and X_f. The last assertion of the lemma follows from the fact that simple functions are dense in $L_p S(f), \Sigma_f, |f|^p d\mu)$. Q.E.D.

REMARK. The idea of considering the operator Q_f is found in T. Ando [1].

LEMMA 5. *Let (Ω, Σ, μ) be a general measure space and P a contractive projection in $L_p(\Omega, \Sigma, \mu)$; $1 \leqq p \neq 2 < \infty$. Then the collection of sets*

$$\mathscr{E} = \{e \mid \text{There exists } f \in L_p(\Omega, \Sigma, \mu);\ Pf = f \text{ such that } S(f) = e\}$$

is a sub σ-ring of Σ (sometimes without maximal element). Furthermore, for $f = Pf$ and $e = S(f)$ a subset $e_0 \subset e$ belongs to $\mathscr{E}$ if and only if $e_0 \in \Sigma_f$ (defined in the previous lemma) and then $P(f\chi_{e_0}) = f\chi_{e_0}$.

Proof. Let $f_i \in L_p(\Omega, \Sigma, \mu)$; $i = 1,2$ be such that $Pf_i = f_i$ and $e_i = S(f_i)$. If $e = e_1 \cup e_2$ then obviously (e, Σ_e, μ) is a σ-finite measure space and hence there exists at least one number α for which $S(f_1 + \alpha f_2) = e$. Set $f = f_1 + \alpha f_2$ and find h_i such that $f_i = h_i f$; $i = 1,2$. Using the operator Q_f defined in the proof of the previous lemma we have $Q_f h_i = f_i/f$ and since Q_f is a conditional expectation relative to Σ_f $e_i = S(f_i) = S(f_i/f) = S(Q_f h_i) \in \Sigma_f$; $i = 1,2$. Thus $e_1 \cup e_2, e_1 - e_2 \in \Sigma_f$ which implies $Q_f \chi_{e_1 \cap e_2} = \chi_{e_1 \cap e_2}$. and $Q_f \chi_{e_1 - e_2} = \chi_{e_1 - e_2}$. Then the definitionof Q_f leads to $P(\chi_{e_1 \cap e_2} f) = \chi_{e_1 \cap e_2} f$ and $P(\chi_{e_1 - e_2} f) = \chi_{e_1 - e_2} f$ i.e. $e_1 \cap e_2$, $e_1 - e_2 \in \mathscr{E}$.

Now consider $e = \bigcup_{n=1}^{\infty} e_n$; $e_n = S(f_n)$; $f_n = Pf_n$; $n = 1, 2, \cdots$. In view of the first part of the proof we can assume that e_n are disjoint. Then $f(\omega) = \sum_{n=1}^{\infty} f_n(\omega)/2^n \|f_n\|$ satisfies $Pf = f$ and $e = S(f)$ i.e. $e \in \mathscr{E}$.

In order to prove the last statement let us observe that if $e_0 \in \mathscr{E}$; $e_0 = S(f_0)$; $f_0 = Pf_0$ and h_0 has been chosen such that $f_0 = h_0 f$ then $Q_f h_0 = f_0/f$ and $S(f_0) = S(Q_f h_0) \in \Sigma_f$. Therefore $Q_f \chi_{e_0} = \chi_{e_0}$ i.e. $P(\chi_{e_0} f) = \chi_{e_0} f$. Conversely, if $e_0 \in \Sigma_f$ then $Q_f \chi_{e_0} = \chi_{e_0}$ and again $P(\chi_{e_0} f) = \chi_{e_0} f$ which implies $S(\chi_{e_0} f) = e_0 \in \mathscr{E}$.

Q.E.D.

THEOREM 6. *Let (Ω, Σ, μ) be a general measure space and P a contractive projection in $L_p(\Omega, \Sigma, \mu)$; $1 \leqq p < +\infty$. Then the range of P is isometric to an L_p-space.*

Proof. Since the theorem is trivial for $p = 2$ we can assume with no loss of generality that $p \neq 2$. Preserving the notation used in the previous lemmas we will consider the family of collections of disjoint sets from $\mathscr{E}$ and order it by inclusion. By the lemma of Zorn this family has a maximal element $\{e_\gamma\}$; $e_\gamma \in \mathscr{E}$ and there exist $f_\gamma \in L_p(\Omega, \Sigma, \mu)$ such that $Pf_\gamma = f_\gamma$ and $e_\gamma = S(f_\gamma)$. Obviously, for each $e \in \mathscr{E}$ $\mu(e \cap e_\gamma) = 0$ for every γ with the exception of countable set $\{\gamma_n\}$. The maximality of the collection $\{e_\gamma\}$ and Lemma 5 will imply that $e = \bigcup_{n=1}^{\infty} (e \cap e_{\gamma_n})$ and this decomposition is unique. Define

$$g_e(\omega) = \begin{cases} 0 & \text{if } \omega \notin e \\ \dfrac{f_{\gamma_n}(\omega)}{2^n \|f_{\gamma_n}\|} & \text{if } \omega \in e \cap e_{\gamma_n}. \end{cases}$$

Evidently, $g_e \in L_p(\Omega, \Sigma, \mu)$, $S(g_e) = e$ and $Pg_e = g_e$. Indeed, by Lemma $e \cap e_{\gamma_n} \in \mathscr{E}$ and $P(f_{\gamma_n}\chi_{e \cap e\gamma_n}) = f_{\gamma_n}\chi_{e \cap e\gamma_n}$ which gives $Pg_e = g_e$.

Let $\{\delta_i\}_{i=1}^{m}$ be a partition of e with sets $\delta_i \in \Sigma_{g_e}$ (defined by Lemma 4) and $X(g_e, \{\delta_i\})$ the finite dimensional subspace spanned by the vectors $\chi_{\delta_i} g_e$. By the last part of Lemma 4

$$X_{g_e} = \overline{\cup X(g_e, \{\delta_i\})}$$

where the union is taken over all possible partitions of e with sets $\delta_i \in \Sigma_{g_e}$. Since every f belonging to the range of P can be written as hg_e with $e = S(f)$ we obtain $f \in X_{g_e}$ and further

$$PL_p(\Omega, \Sigma, \mu) = \overline{\cup X(g_e, \{\delta_i\})}$$

where the union is taken over all sets $e \in \mathscr{E}$ and partitions $\{\delta_i\}$ of e with sets from Σ_{g_e}.

Next we shall show that finite dimensional subspaces $X(g_e, \{\delta_i\})$ form a net. Indeed, let us prove that $X(g_{e_1}, \{\delta_i\})$ and $X(g_{e_2}, \{\sigma_j\})$ are contained in $X(g_e, \{\eta_k\})$ where $e = e_1 \cup e_2$ and $\{\eta_k\}$ is the common refinement of $\{\delta_i\}$ and $\{\sigma_j\}$. Since $\delta_i \in \Sigma_{g_{e_1}}$, it follows from Lemma 5 that $\delta_i \in \Sigma_{g_e}$ and in the same way $\sigma_j \in \Sigma_{g_e}$. Thus their common refinement $\{\eta_k\}$ is a partition of e with sets from Σ_{g_e}. Furthermore, for any $\omega \in \eta_k$ $\omega \in e \cup e_{\alpha_{n_0}}$ (since $e = \bigcup_{n=1}^{\infty} (e \cap e_{\alpha_n})$) and $\omega \in e_1 \cap e_{\beta_{m_0}}$ (since $e_1 = \bigcup_{m=1}^{\infty} (e_1 \cap e_{\beta_m})$). Consequently, $e_{\alpha_{n_0}} = e_{\beta_{m_0}} = e_\gamma$ and

$$\chi_{\eta_k}(\omega)g_e(\omega) = \frac{f_\gamma(\omega)}{2^{n_0}\|f_\gamma\|}\,;\ \chi_{\eta_k}(\omega)g_{e_1}(\omega) = \frac{f_\gamma(\omega)}{2^{m_0}\|f_\gamma\|}.$$

We have to mention that e_{α_n}, e_{β_m} and e_γ are sets from the maximal collection defined in the beginning of this proof.

In conclusion the range of P can be expressed as the closure of the union of a net of finite dimensional subspaces $X(g_e, \{\delta_i\})$, each of these isometric to l_p^n; $(n = \dim X(g_e, \{\delta_i\})))$; the space of sequences $(c_1, \cdots, c_n)$ with the norm $(\sum_{i=1}^n |c_i|^p)^{1/p}$ (by Lemma 4 $X(g_e, \{\delta_i\})$ is isometric to the subspace of $L_p(e, \Sigma_{g_e}, |g_e|^p d\mu)$ generated by $\chi_{\delta_i} \in \Sigma_{g_e}$; $i = 1, \cdots, n$). Such a space is called $N_{1,p}$ and it has been proved recently by M. Zippin [6] Theorem 11.6 that a Banach space X is $N_{1,p}$ if and only if it is isometric to an L_p-space. This completes the proof. Q.E.D.

According to J. Lindenstrauss and A. Pełczyński [5], Definition 3.1, a Banach space X is called an $\mathscr{L}_{p,\lambda}$; $1 \leqq p \leqq +\infty$, $1 \leqq \lambda < +\infty$, if for every finite dimensional subspace B of X there exists another finite dimensional subspace E of X containing B and a linear operator τ_E from E onto l_p^n; $n = \dim E$, such that $\|\tau_B\| \, \|\tau_B^{-1}\| \leqq \lambda$. They have proved in [5] Theorem 7.1 that for every $\mathscr{L}_{p,\lambda}$ space X; $1 < p < \infty$ there exists an an L_p-space Y and a projection P in Y having the norm $\|P\| \leqq \lambda$ whose range is isometric to X. Using this fact and Theorem 6 we obtain the following result.

COROLLARY 7. *A Banach space X is isometric to an L_p-space*; $1 \leqq p < +\infty$ *if and only if it is an $\mathscr{L}_{p,\lambda}$ space for every $\lambda > 1$.*

Proof. For $p = 1$ the proof has been given in [5] Corollary 5 of Theorem 7.1. For $1 < p < \infty$ it follows from the fact already remarked that X is isometric to the range of a contractive projection in an L_p-space and thus, by Theorem 6, X itself is isometric to an L_p-space. Q.E.D.

REMARK. J. Lindenstrauss and A. Pełczyński [5], Corollary 4 of Theorem 7.1 proved Corollary 7 under the assumption that X is separable, using instead of Theorem 6, the weaker version given by T. Ando [1], Theorem 4 for L_p-spaces over finite measure spaces.

ACKNOWLEDGEMENT

The author wishes to express his gratitude to S. R. Foguel for the very helpful conversations regarding this paper.

REFERENCES

1. T. Ando, *Constractive projections in L_p-spaces*, Pacific J. Math. **17** (1966), 391–405.

2. R. G. Douglas, *Contractive projections on an $\mathscr{L}_1$-space*, Pacific J. Math. **15** (1965), 443–462.

3. N. Dunford and J. Schwartz, *Linear Operators* I, Interscience Publishers, New York, 1958.

4. A. Grothendieck, *Une caractérisation vectorielle métrique des espaces L^1*, Cad. J. Math. 7 (1955), 552–561.

5. J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in L_p-spaces and their applications*, Studia Math. **29** (1968), 275–326.

6. M. Zippin, *On bases in Banach spaces*, Ph.D. Thesis, Hebrew University of Jerusalem, 1968 (Hebrew with English summary).

UNIVERSITY OF MINNESOTA